CHAPTER 13 The Driven, Damped Weight on a Spring: One of the Most Important Problems in Physical Science

Everything we perceive about our surroundings is based on the *forces* that connect us to those surroundings. The content of physics might be defined as a study of the origin and consequences of forces in nature. As for any quantity that varies with distance, one can in general expand the force (function) in a power series about some convenient point in space, say, the origin:

$$\text{Net force} = m(d^2x/dt^2) = A_0 + A_1x + A_2x^2 + \cdots \qquad (13\text{-}1)$$

in which x represents the displacement from some arbitrary origin (say, the center of an atom). The first thing to recognize is that even a relatively simple force, such as the Coulomb force between two charged particles, $m(d^2x/dt^2) = qq'/x^2$, may be represented by an *infinite* number of terms in the power series of Eq. 13-1. Fortunately, one can understand much of what is presently known about common forces by making two simplifications in Eq. 13-1.

First, we can often dispense with the constant term, A_0, by choosing a suitable "reference" frame. For example, we successfully neglected the (constant) force of gravity when we considered electrophoresis experiments. Second, if the force is so *weak* that the observed displacement, x, is very *small,* then we may neglect all the higher order terms, leaving

$$\text{Net force} = m(d^2x/dt^2) = A_1x \tag{13-2}$$

Finally, if the constant, A_1, is now re-labeled as a new constant, $-k$, then Eq. 13-2 is immediately recognized as the equation for the (restoring) force for a weight (of mass, m) on a spring:

$$\text{Net force} = \boxed{m(d^2x/dt^2) = -kx} \tag{13-3}$$

For example, even though we may know that the force binding an electron to an atom or molecule is a *Coulomb* force, we may still treat the electrons in a substance as if the electrons were bound to their respective atoms by *springs,* provided that whatever forces we apply do not displace the electrons very far from their equilibrium positions. Thus, because the electric and magnetic forces from electromagnetic waves (light, X rays, radiofre-

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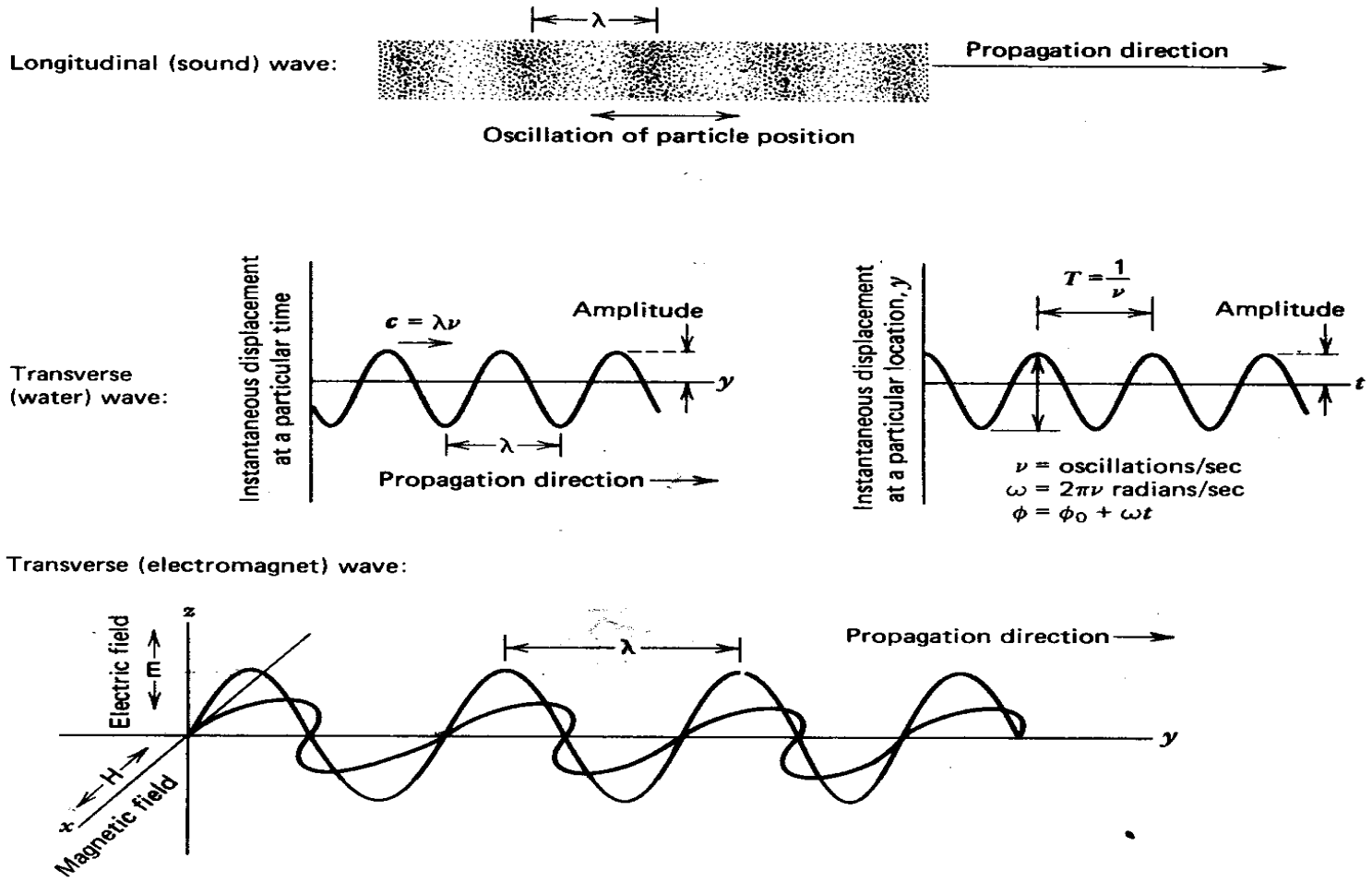

FIGURE 13-1. Schematic diagrams of a longitudinal (sound) wave, a transverse (water) wave, and a transverse (electromagnetic) wave. In each case, the wave is taken to be monochromatic (see text). The electromagnetic wave is also taken to be plane-polarized (see text), for simplest display. All waves are generated continuously at the left of the figure, so that propagation is to the right. The parameters associated with the wave motion are described below. It will prove useful in later discussion to switch back and forth between the "snapshot" picture of wave displacement as a function of propagation distance at a particular time (middle left diagram), and the equivalent alternative picture of wave displacement as a function of time at a given propagation distance (middle right diagram).

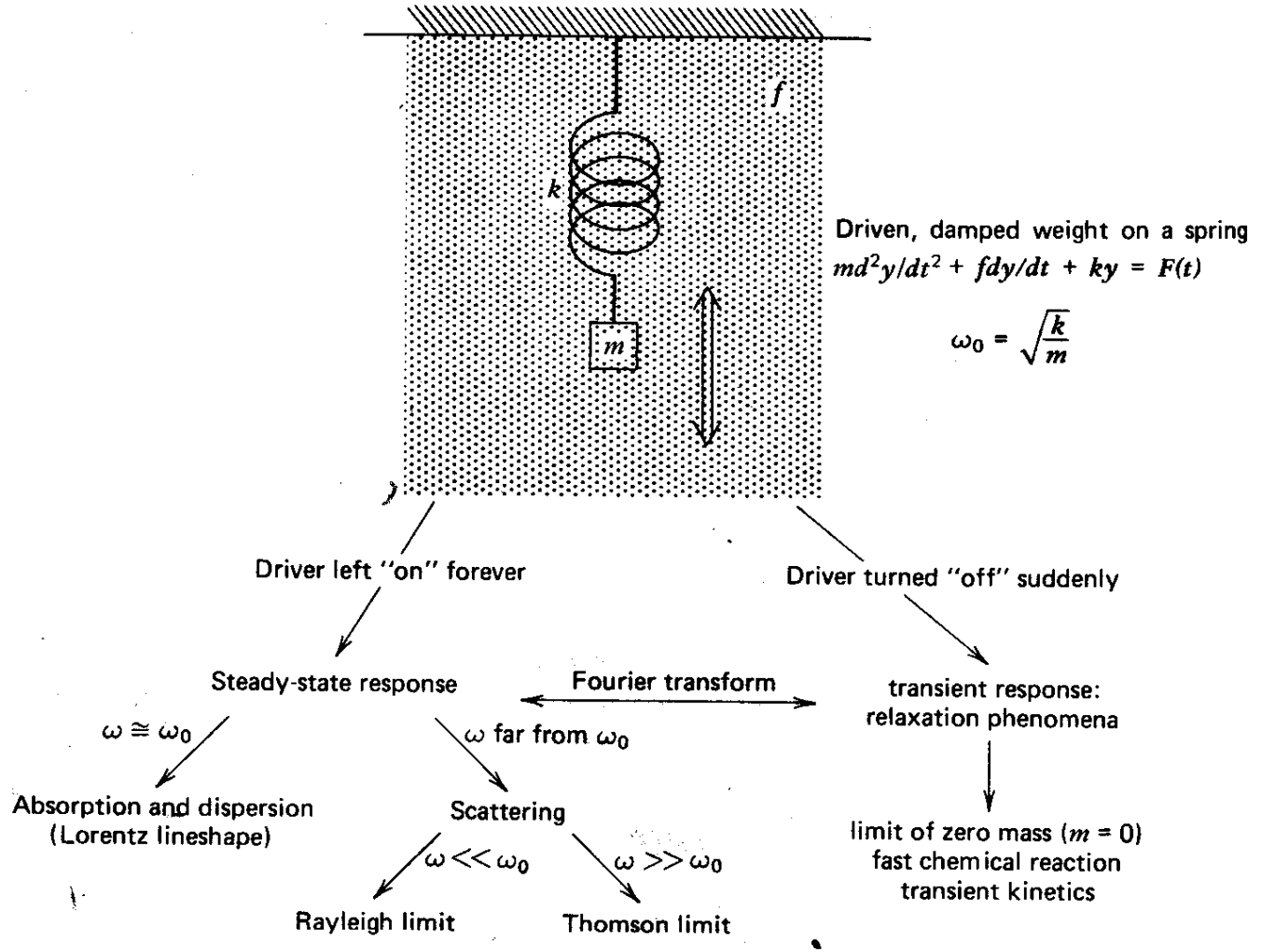

FIGURE 13-4. Interrelations between various scattering, spectroscopy, and transient experiments, based on analogy to the motion of a sinusoidally driven (or suddenly displaced), damped weight on a spring. The weight has mass, m, bound to a spring of force constant, k, immersed in a medium to give frictional coefficient, f, and subjected either to a sudden displacement or to a sinusoidally time-varying force, $F(t) = F_0\cos(\omega t)$. Steady-state phenomena are discussed in Chapters 14 and 15, transient phenomena in Chapter 16, and Fourier methods in Section 6.

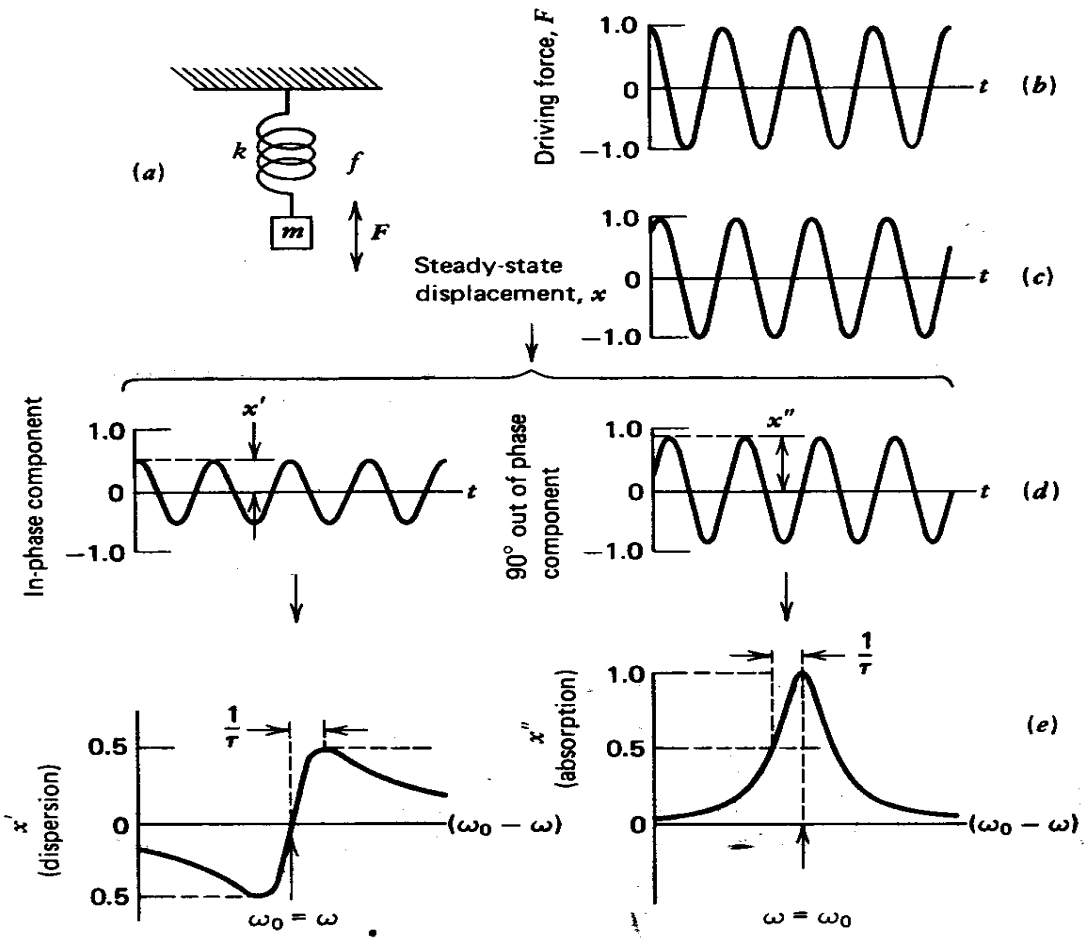

FIGURE 13-5. Dispersion and absorption, as derived from the steady-state response of a driven, damped weight on a spring. When subjected to a sinusoidally time-varying driving force (*b* in the figure), the mass on a damped spring (Fig. 13-5*a*) will eventually settle into a steady-state oscillation at the *same frequency* as the driver, but with *different phase* (see Fig. 13-5*c*). The magnitude of the displacement (Fig. 13-5*c*) may be decomposed into one component that is exactly *in-phase* with the driver (Fig. 13-5*d*, left diagram) and a component exactly *90°-out-of-phase* with the driver (Fig. 13-5*d*, right diagram). The variation with (driving) frequency of the in-phase amplitude is called the "dispersion" spectrum (Fig. 13-5*e*, left diagram) and the frequency-variation of the 90°-out-of-phase amplitude is called the "absorption" spectrum. Applications include particles from scattering and spectroscopy. $\omega_0 = (k/m)^{1/2}$, and $(1/\tau) = f/2m$ in the mechanical analog.

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$$F = F_0 \cos(\omega t) \tag{13-14}$$

and subject to a frictional (damping) force proportional to the velocity of the moving weight, is given by:

$$\boxed{m\, d^2x/dt^2 = \text{net force} = -k\, x - f\, dx/dt + F_0 \cos(\omega t)} \tag{13-15}$$

Here $-kx$ is the restoring force from the spring, $-f\, dx/dt$ is the frictional force opposing the motion, and $F_0 \cos(\omega t)$ is the applied driving force. It is left to the reader to verify that Eq. 13-16 is a solution of Eq. 13-15, where Eq. 13-12, $\omega_0 = \sqrt{(k/m)}$, has been used to simplify the final expression:

$$x = x' \cos(\omega t) + x'' \sin(\omega t) \tag{13-16a}$$

where

GENERAL CASE

$$x' = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + f^2\omega^2} \tag{13-16b}$$

and

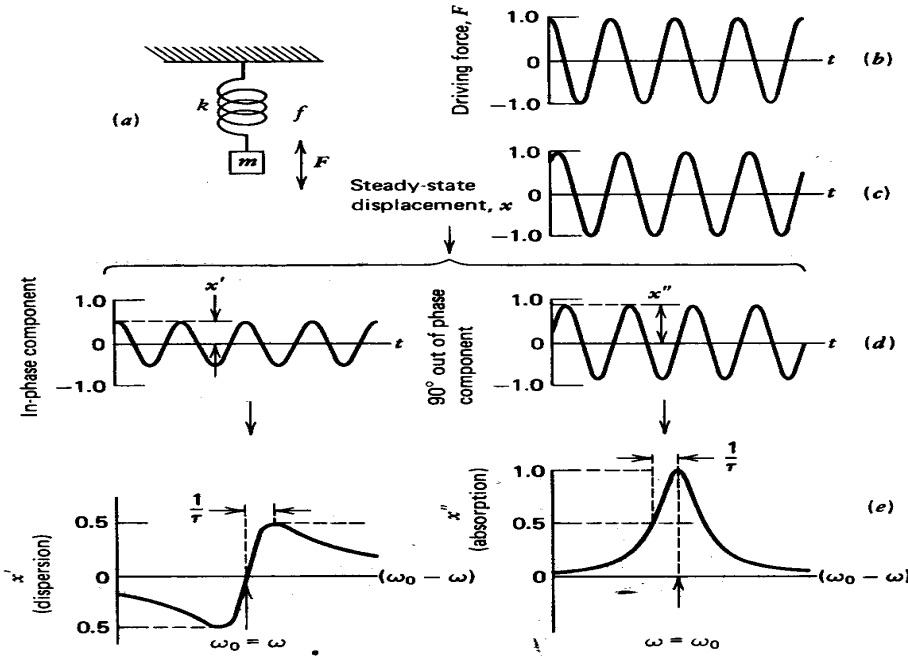
$$x'' = F_0 \frac{f\, \omega}{m^2(\omega_0^2 - \omega^2)^2 + f^2\omega^2} \tag{13-16c}$$

Equation 13-16 simply states mathematically what is shown pictorially in Fig. 13-5, namely that the steady-state displacement, x, may be broken down into two components, $x' \cos(\omega t)$ and $x'' \sin(\omega t)$, which may be thought of as the parts of the response either exactly *in-phase* with the driver [i.e., vary in time as $\cos(\omega t)$], or exactly *90°-out-of-phase* with the driver

$$\text{Dispersion} = \lim_{\frac{|\omega_0-\omega|}{\omega_0+\omega}<<1} (x') = \frac{F_0}{2m\omega_0}\left[\frac{(\omega_0-\omega)\tau^2}{1+(\omega_0-\omega)^2\tau^2}\right] \quad \text{(13-36a)}$$

(spectroscopy)

$$\text{Absorption} = \lim_{\frac{|\omega_0-\omega|}{\omega_0+\omega}<<1} (x'') = \frac{F_0}{2m\omega_0}\left[\frac{\tau}{1+(\omega_0-\omega)^2\tau^2}\right] \quad \text{(13-36b)}$$



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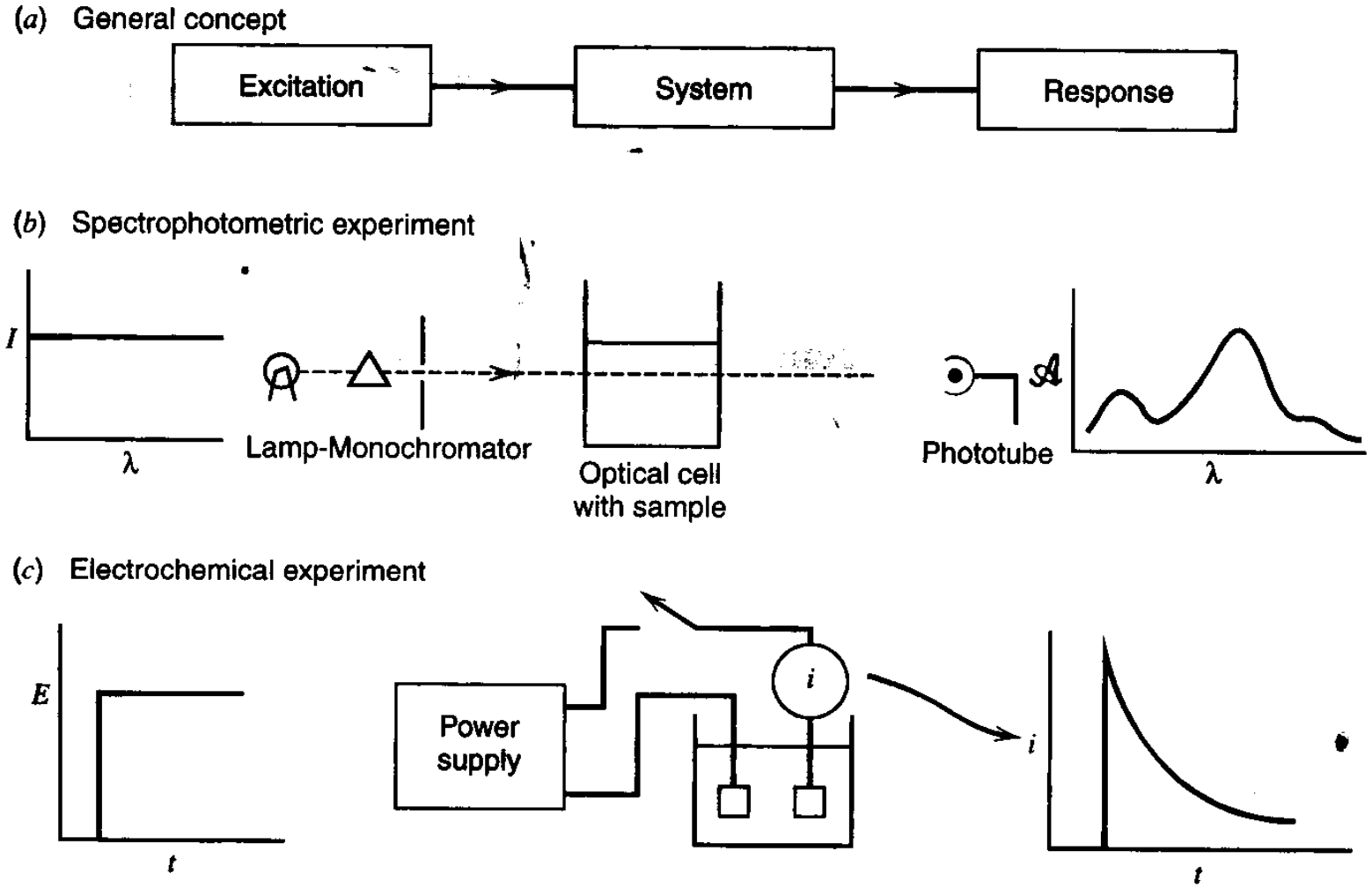


Figure 1.3.3 (*a*) General principle of studying a system by application of an excitation (or perturbation) and observation of response. (*b*) In a spectrophotometric experiment, the excitation is light of different wavelengths (λ), and the response is the absorbance ($\mathcal{A}$) curve. (*c*) In an electrochemical (potential step) experiment, the excitation is the application of a potential step, and the response is the observed *i*-*t* curve.

13.D. ZERO-MASS ON A DAMPED SPRING: RELAXATION PHENOMENA

A final limiting situation is the case in which $m = 0$ (massless "weight" on a spring). The equation of motion can now be thought of as:

$$f\,dx/dt + kx = F_0 \cos(\omega t) \qquad \text{Steady-state experiment} \qquad (13\text{-}42)$$

or

$$f\,dx/dt + kx = 0 \qquad \text{Transient experiment} \qquad (13\text{-}42)$$

The solutions to Equations 13-42 and 13-43 can be verified (see Problems) as

$$x = x' \cos(\omega t) + x'' \sin(\omega t) \qquad (13\text{-}44)$$

where

$$x' = F_0 \frac{k}{k^2 + f^2\omega^2} \qquad (13\text{-}45a)$$

and

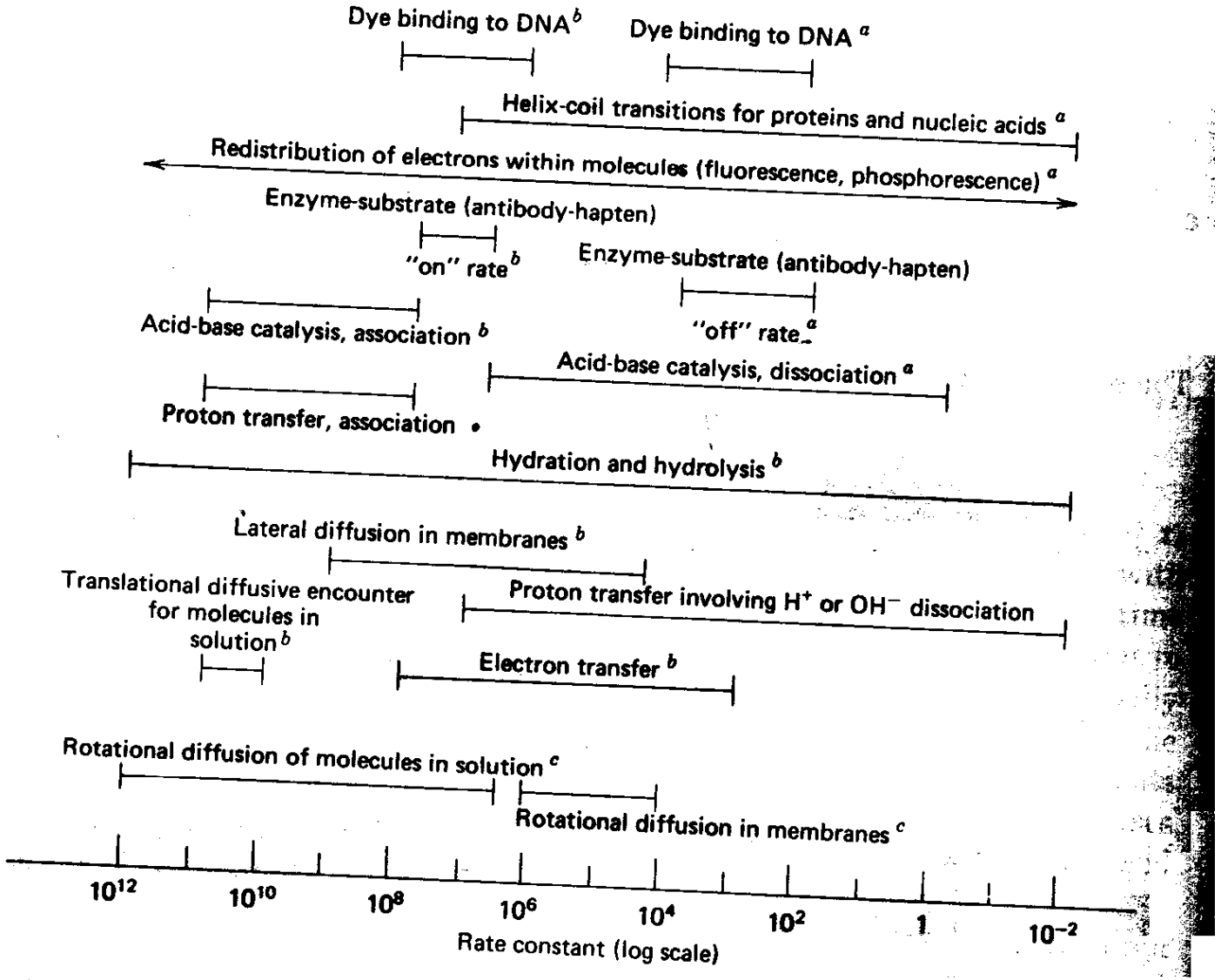
$$x'' = F_0 \frac{f\omega}{k^2 + f^2\omega^2} \qquad (13\text{-}45b)$$

Steady-state

or

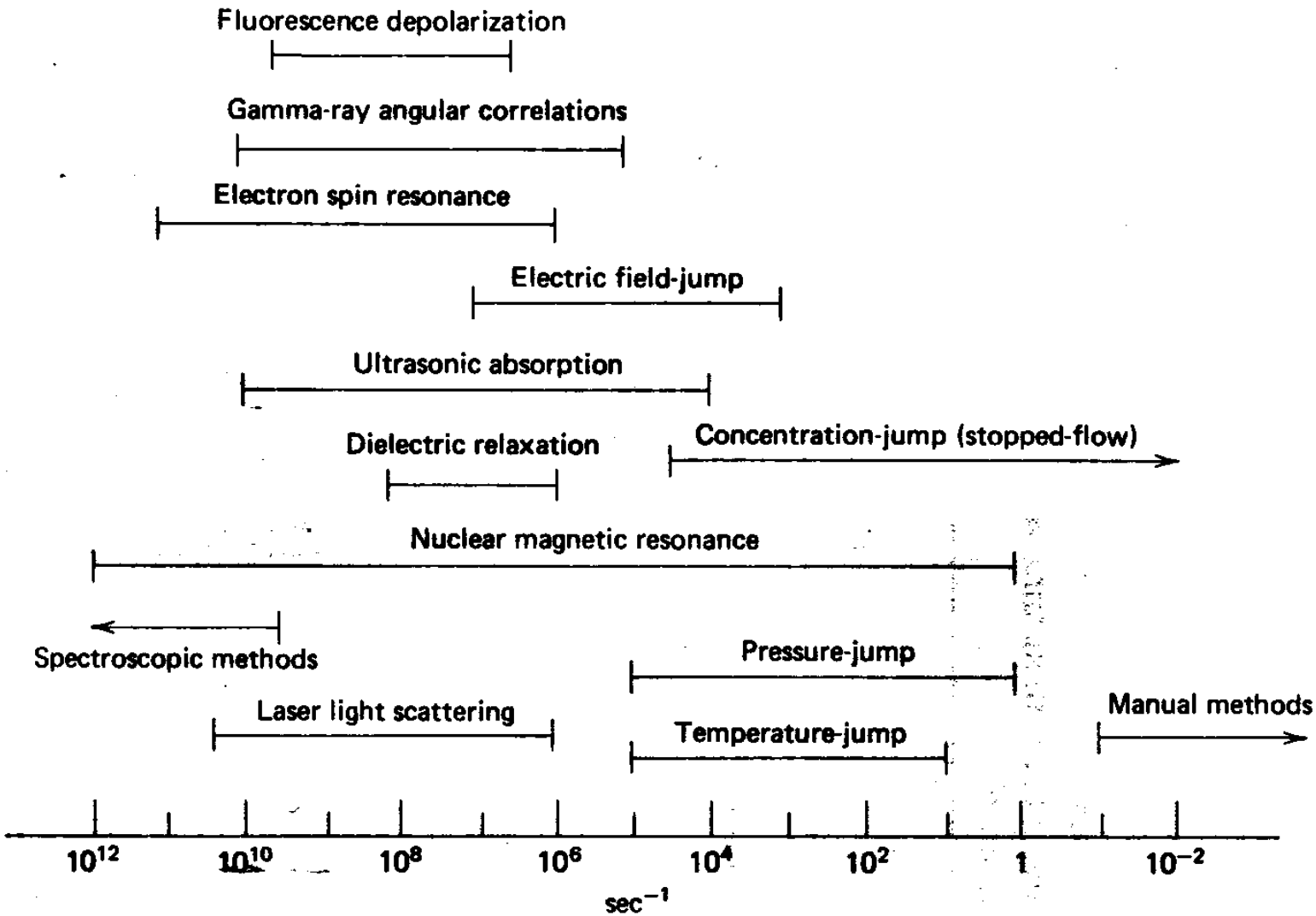
$$x = x_0 \exp[-kt/f] \qquad \text{Transient} \qquad (13\text{-}46)$$

Table 13-2. Schematic diagram of rate constants for various chemical and physical processes (typical rates are shown).



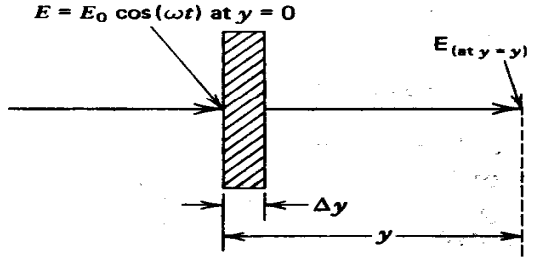
a rate constant = (1/lifetime) for first-order process, b rate constant = 2nd-order rate constant in M^{-1} *sec*$^{-1}$; *c rate constant = (1/(rotational correlation time)).*

Table 13-3. Schematic diagram of the range of rates typically accessible by means of each of the techniques listed.



14.A. ABSORPTION AND REFRACTIVE INDEX: BASIS FOR SPECTROSCOPY AND MICROSCOPY

Suppose that monochromatic radiation is directed through a thin slab of matter of thickness, Δy (see diagram below). In this section, we are interested in the electric field amplitude of the emerging wave, at a distance y from the slab. First, since the observed electric field at y was produced



by oscillating electrons in the slab at a time $(t - (y/c))$ ago, where c is the speed of light (in a vacuum), it is readily seen that even if the slab had no effect whatever on the incident radiation, the electric field at y would be of the form

$$E_{(\text{at } y)} = E_0 \cos\left[\omega\left(t - \frac{y}{c}\right)\right] \tag{13-9}$$

$$= \text{Re}\left[E_0 \exp\left[i\omega\left(t - \frac{y}{c}\right)\right]\right] \tag{14-1}$$

However, if the wave slows down while passing through the slab, then from the usual definition of refractive index, n'

$$\boxed{\text{Refractive index} = n' = \frac{\text{(speed of light in vacuo)}}{\text{(speed of light in slab)}}} \tag{14-2}$$

the emerging wave will have been slowed down by a time, $t = (n' - 1)(\Delta y)/c$, so that the electric field at y now becomes

$$E_{(\text{at } y)} = E_0 \cos\left[\omega\left(t - \frac{(n' - 1)(\Delta y)}{c} - \frac{y}{c}\right)\right] \tag{14-3a}$$

* *Throughout this section, the results are displayed in both real and complex notation where appropriate to familiarize the reader with the intuitively understandable ("real") description as well as with the mathematically compact ("complex") format.*

$$= \mathrm{Re}\left[E_0 \exp\left[\left(i\omega\left(t - \frac{(n'-1)(\Delta y)}{c} - \frac{y}{c}\right)\right]\right]$$

$$= \mathrm{Re}\Big[\underbrace{\exp[-i\omega(n'-1)(\Delta y)/c]}_{\text{Effect of refractive index} \neq 1} \underbrace{E_0 \exp\left[i\omega\left(t - \frac{y}{c}\right)\right]}_{\text{Electric field of wave with no slab present}}\Big] \qquad (14\text{-}3b)$$

Equation 14-3b shows that with complex notation it is easy to separate the behavior of the wave in the absence of any refraction from a term that consists solely of the effect due to slowing down of the radiation while passing through the matter.

In addition to the reduction in *velocity* (Eq. 14-3) while passing through the slab, there will be a *loss. dI, in intensity, I,* on passing through the slab. For a sufficiently thin slab, we would expect the rate of decrease in intensity to be proportional to the intensity, to the thickness of the slab, *dy,* and to the concentration of absorbing molecules, m (in moles/liter), as in Eq. 14-4.

$$dI = -kImdy \qquad (14\text{-}4)$$

in which k is a proportionality constant characteristic of the molecule in question. Equation 14-4 in its integrated form is known as "Beer's Law":

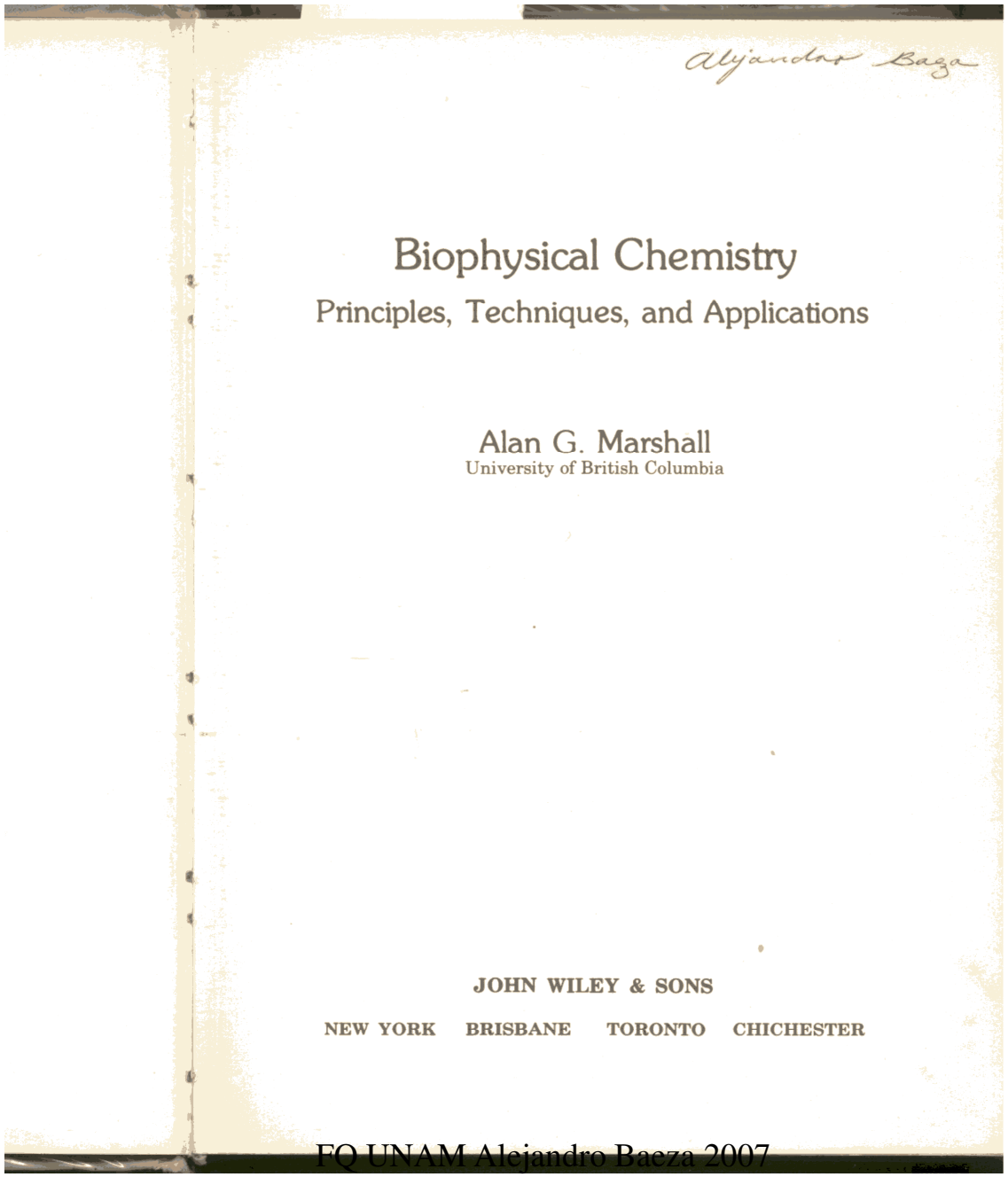
$$\log_e(I/I_0) = -km\ \Delta y$$

$$\boxed{\begin{aligned}\text{"Transmittance"} = I/I_0 &= 10^{-(km\Delta y/2.303)} = 10^{-(\epsilon m \Delta y)} \\ &= 10^{-A} = e^{-(2.303\epsilon\Delta y N/N_0)}\end{aligned}} \qquad (14\text{-}5)$$

where ϵ is called the "molar absorbancy index" (formerly called the "molar extinction coefficient"), I_0 is the intensity incident on the slab, and A is called the "absorbance," or "optical density," N is the number of molecules per cm^3, and N_0 is Avogadro's number. Thus, since electromagnetic radiation intensity is proportional to the square of electric field amplitude of the transmitted wave, and since $\sqrt{e^{-x}} = (e^{-x})^{1/2} = e^{-(x/2)}$, Eq. 14-3 can now be extended to incorporate the absorption of energy described by Eq. 14-5:

$$E_{(\text{at } y)} = \exp[-2{,}303\epsilon\, N(\Delta y)/2N_0]\, E_0 \cos\left[\omega\left(t - \frac{(n'-1)\Delta y}{c} - \frac{y}{c}\right)\right] \qquad (14\text{-}6a)$$

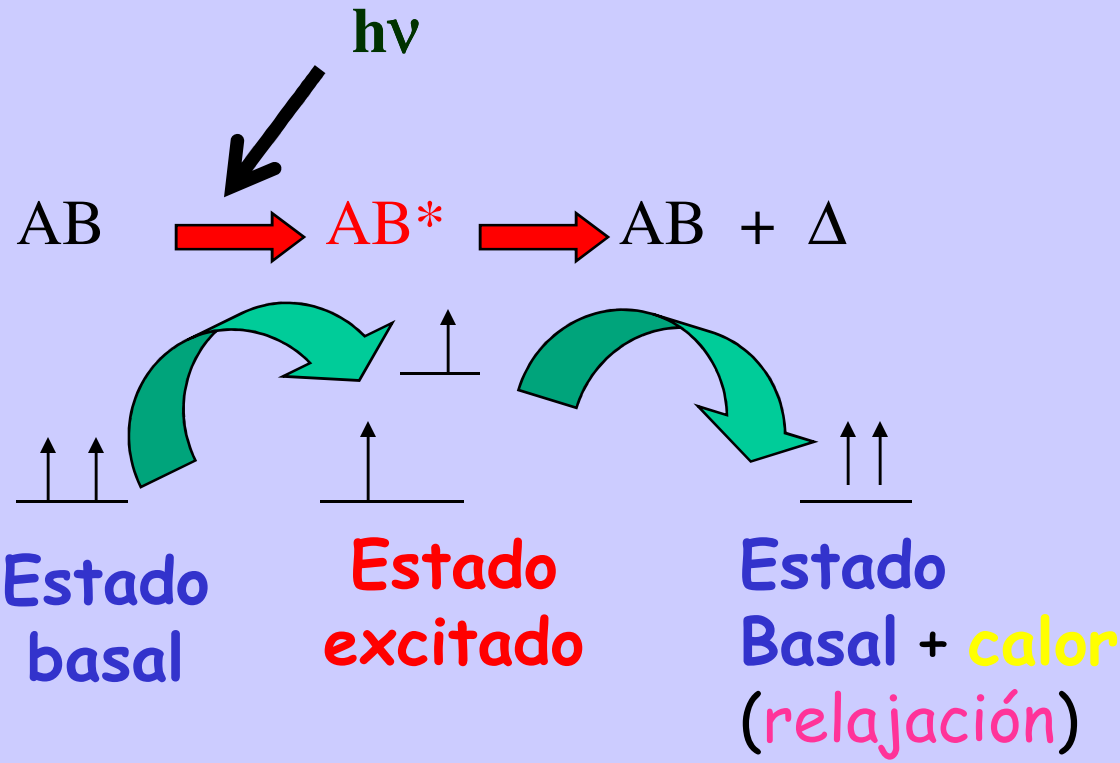
$$= \mathrm{Re}\Big[\underbrace{\exp[-2{,}303\epsilon\, N\Delta y/2N_0]}_{\text{Effect of finite absorption}} \underbrace{\exp[-i\omega(n'-1)\Delta y/c]}_{\text{Effect of refractive index} \neq 1.0} \underbrace{E_0 \exp\left[i\omega\left(t - \frac{y}{c}\right)\right]}_{\text{Electric field of wave with no slab present}}\Big] \qquad (14\text{-}6b)$$


Alejandro Baeza
Biophysical Chemistry
Principles, Techniques, and Applications
Alan G. Marshall
University of British Columbia
JOHN WILEY & SONS
NEW YORK BRISBANE TORONTO CHICHESTER

Interacciones energía electromagnética-materia:

Dispersión
Refracción
Transmisión
Fluorescencia
Polarización

Absorción:



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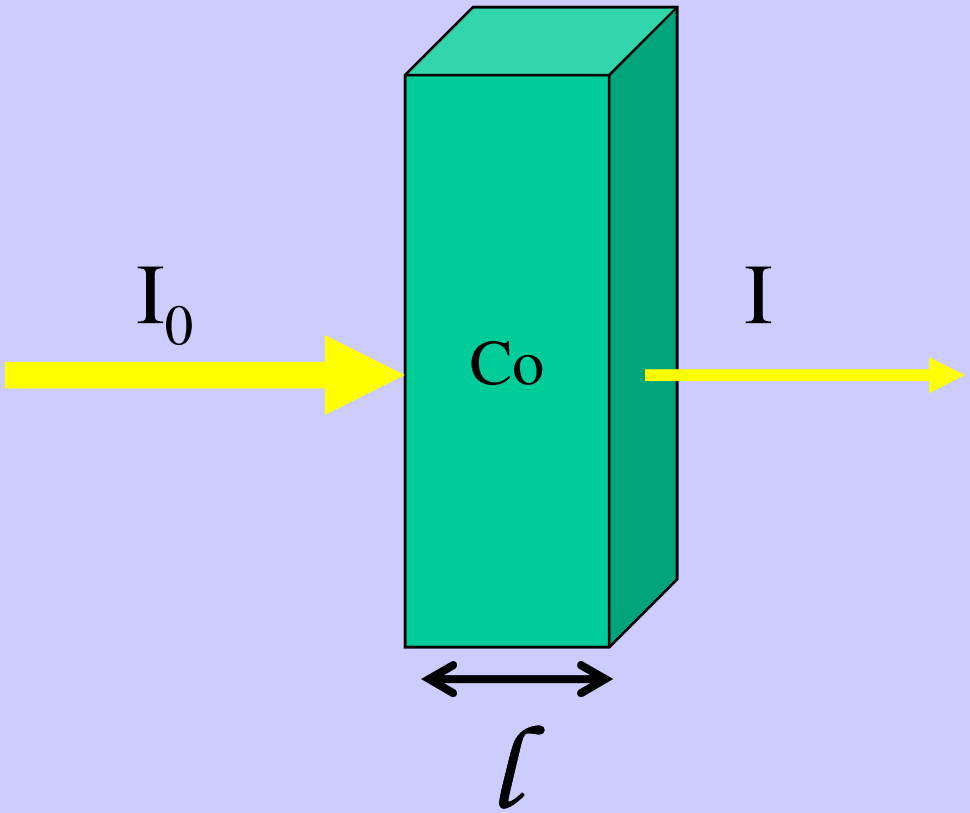
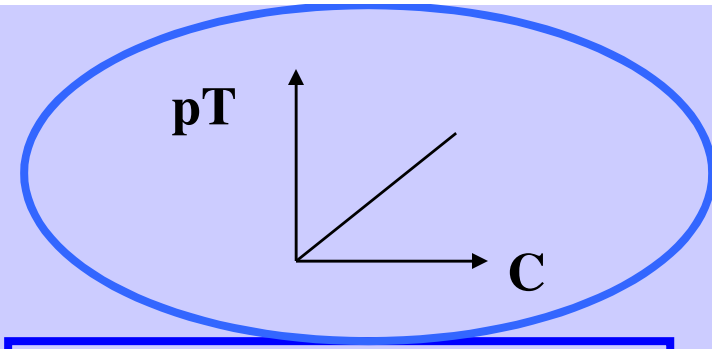
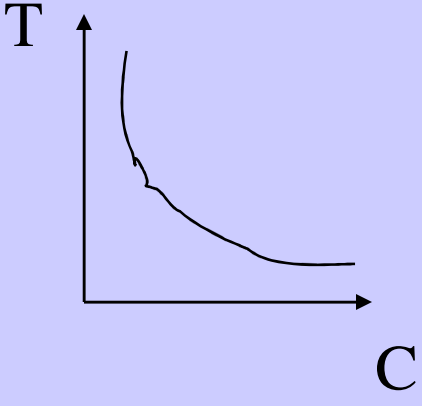




-logT = pT = A = εlCo

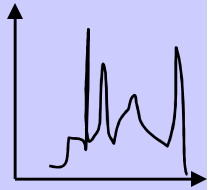


$$(I/I_0) = T = 10^{-\varepsilon l Co}$$

absorción

absorciometría

Espectroscopía



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