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Analogies of transport properties

Behind the development of the Reynolds and Chilton-Colburn analogies is the appreciation that there are certain similarities among the transport of momentum, mass, and energy. *Transport phenomena* is the integrated study of these three physical properties—they intertwine under many circumstances. Generally, since concentration and temperature are scalar quantities, analogies between mass and heat transport are more valid than those with momentum.

Nevertheless, with proper definitions, the transports of these quantities all depend on transport coefficients with the units of [length^2/time]—*kinematic viscosity*, *diffusivity*, and *thermal diffusivity*. This is why kinematic viscosity is also called momentum diffusivity. The flux laws of these quantities—*Newton's law of viscosity*, *Fick's law*, and *Fourier's law*—all share the same linear constitutive equation. Certainly, we may also identify similar features in their conservation equations. The idea is that, say, when we solve a diffusion problem, we may want to find the solution of a similar heat conduction problem. The main motivation underlying the establishment of transport analogies, however, is that we can take experimental measurements or correlations in one system, and apply it to another. For example, we can apply heat transfer data to a mass transport problem where experimentation with mass transport may be difficult. This is particularly important with turbulent flow. (You'd see more advanced analogies in a graduate level course.) For now, keep the following table handy.

Table 1. Analogous quantities in transport phenomena

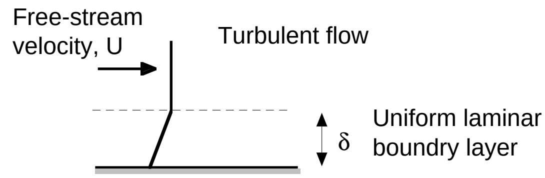
	Momentum	Mass	Energy
Transport quantity per volume	ρu_x	C_A	$\rho C_p T$
Transport coefficient	μ [g·cm^{-1}s^{-1}]	D_{AB} [cm^2·s^{-1}]	k [cal·cm^{-1}s^{-1}K^{-1}]
Diffusivity [cm^2·s^{-1}]	$\nu = \mu/\rho$	D_{AB}	$\alpha = k/\rho C_p$
Flux law	$\tau_{xz} = -\nu \frac{d}{dz}(\rho u_x)$	$J_z = -D_{AB}\frac{d}{dz}C_A$	$q_z = -\alpha\frac{d}{dz}(\rho C_p T)$
Dimensionless transport groups	$Re = \frac{UL}{\nu}$	$Sc = \frac{\nu}{D_{AB}}$	$Pr = \frac{\nu}{\alpha}\left(= \frac{C_p\mu}{k}\right)$
Other dimensionless groups used in transport correlations		$Nu_m = \frac{k_c L}{D_{AB}}$	$Nu_h = \frac{h L}{\rho C_p \alpha}$
		$Pe = ReSc = \frac{UL}{D_{AB}}$	$Pe = RePr = \frac{UL}{\alpha}$
		$St = \frac{Nu_m}{ReSc} = \frac{k_c}{U}$	$St = \frac{Nu_h}{RePr} = \frac{h}{\rho C_p U}$
For turbulent transport:			
Reynolds analogy (Sc = Pr = 1)	$\frac{f}{2}$	$\frac{k_c}{U}$	$\frac{h}{\rho C_p U}$
Chilton-Colburn j-factor	$\frac{f}{2}$	$\frac{k_c}{U}Sc^{2/3} = \frac{Nu_m}{Re\,Sc^{1/3}}$	$\frac{h}{\rho C_p U}Pr^{2/3} = \frac{Nu_h}{Re\,Pr^{1/3}}$

Note: The cgs unit of viscosity is the same as a poise. Strictly speaking, the Fick's law should be defined on the basis on a mole fraction gradient. Nu_m is the same as the Sherwoord number (**Sh**).
Also, we may find the Lewis number defined as **Le** = Pr/Sc = D_{AB}/α.

Reynolds Analogy

Here are the derivation steps to clarify Section 6.2. From fluid dynamics (AMES 103A), the friction factor is defined as

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \qquad (1)$$



Representation of the uniform boundary layer model.

We consider flow over a flat plate. If we assume a simple shear flow in a uniform laminar boundary layer, we can approximate the wall stress as

$$\tau_w \approx \mu \frac{U}{\delta} \qquad (2)$$

where U is the free-stream velocity and δ is fictitious uniform boundary layer thickness. So,

$$\delta \approx \mu \frac{U}{\tau_w} = \frac{\mu U}{\frac{1}{2}\rho U^2 f} = \frac{\nu}{\frac{1}{2} U f} \qquad (3)$$

and dividing by the length of the flat plate, L:

$$\frac{\delta}{L} \approx \frac{\nu}{\frac{1}{2} UL f} = \frac{1}{\frac{1}{2} Re_L f}, \quad \text{where} \quad Re_L = \frac{UL}{\nu} \qquad (4)^{‡}$$

Next, we define the Sherwood number

$$Sh_L = \frac{\bar{k} L}{D_{AB}} \qquad (5)$$

which is based on the length scale L. The average mass transfer coefficient $\bar{k}$ is based on the film model and some concentration boundary layer thickness δ_c:

$$\bar{k} = \frac{D_{AB}}{\delta_c} \qquad (6)$$

Thus

$$Sh_L = \frac{D_{AB}}{\delta_c} \frac{L}{D_{AB}} = \frac{L}{\delta_c} = \frac{L}{\delta} \frac{\delta}{\delta_c} \qquad (7)$$

Finally, we know from scaling analysis that when Sc = 1, $\delta = \delta_c$, and thus the Sherwood number can be evaluated as

$$\frac{\bar{k} L}{D_{AB}} = \frac{L}{\delta} = \frac{\frac{1}{2} UL f}{\nu}$$

If we equate the very first and last terms,

$$\frac{1}{2} f = \frac{\bar{k}}{U} \qquad (8)$$

since $\nu / D_{AB} = 1$. This is the *Reynolds analogy* which relates the friction factor to the mass transfer coefficient.

‡ If we invoke Blasius formula for turbulent flow, the friction factor is $f = 0.0791\ Re^{1/4}$, and we have the functional dependence $\delta/L \sim Re^{-3/4}$.

Chilton-Colburn Analogy

We now take $Sc \neq 1$, and use Eq. (7). Thus we need to say something about the ratio δ/δ_c. From a proper scaling or from a boundary layer analysis, we should find that $\delta/\delta_c \sim Sc^{1/3}$. We use this relation in (7), and further substitute for δ/L by Eq. (4), and we have

$$Sh_L = \frac{1}{2} Re_L\, f\, Sc^{1/3}$$

and rearrangement gives

$$\frac{1}{2} f = \frac{Sh_L}{Re_L Sc^{1/3}} = j_D \qquad (9)$$

which is the *Chilton-Colburn analogy*, and the mass transfer term on the RHS is also called the *j-factor*, j_D.

Note that we can also write

$$\frac{Sh_L}{Re_L Sc^{1/3}} = \frac{\bar{k} L}{D_{AB}} \frac{\nu}{UL} \left(\frac{D_{AB}}{\nu}\right)^{1/3} = \frac{\bar{k}}{U} \left(\frac{\nu}{D_{AB}}\right)^{2/3} \qquad (10)$$

where we can further define $St = \bar{k}/U$ as the Stanton number.